Mathematical Appendix For:
Remittances, Inflation and Exchange Rate Regimes in Small Open Economies

Christopher P. Ball (Quinnipiac University)
Martha Cruz-Zuniga (Catholic University of America)
Claude Lopez (University of Cincinnati)
Javier Reyes (University of Arkansas)

March 2010

The Model

The utility function is assumed separable in all of its components and over time.

$$U\left(c_t^T, c_t^N, m_t\right) = \int_0^\infty \left[\gamma \log(c_t^T) + (1-\gamma)\log(c_t^N) + \alpha \log(m_t)\right] e^{-\rho t} dt \qquad \text{(A.1)}$$

where c_t^T, c_t^N, and $m_t \left(\equiv M_t / E_t \right)$ denote consumption of the traded good and the non-traded good, and real money balances in terms of the traded good, respectively.

Individuals can hold internationally traded assets yielding constant world interest rate, r, earn income from the sale of traded and non-traded goods, receive/give transfers to the government and receive exogenous foreign-currency remittances from abroad. This can be expressed in the following flow budget constraint[1]

$$\dot{a}_t = ra_t + y_t^T + \frac{y_t^N}{e_t} + \tau_t - c_t^T - \frac{c_t^N}{e_t} - i_t m_t + f_t \qquad \text{(A.2)}$$

where y_t^T represents the traded good, y_t^N, the non-traded good, $e_t \left(\equiv E / P_H \right)$, the real exchange rate, a_t, net asset holdings, τ_t, government transfers and, f_t, the value of remittances. The law of one price holds for the tradable goods and the foreign price label for the traded good is equal to one.

For simplicity, total employment is set to unity and l_t represents employment in the traded good sector, leaving $1-l_t$ in the non-traded sector. The production functions in each sector are:

[1] Notice this budget constraint is consolidated. For the full derivation of this, see the last part of this appendix on solving the dynamic systems in state-costate form.

$$y_t^T = A_t l_t^{\alpha} \qquad 0 < \alpha \le 1 \tag{A.3.a}$$

and

$$y_t^N = B_t \left(1 - l_t\right)^{\beta} \qquad 0 < \beta \le 1 \tag{A.3.b}$$

where A_t and B_t represent technology parameters and the production functions are concave.

Individuals maximize (A.1) subject to (A.2), (A.3.a), and (A.3.b). Doing so yields the following optimality conditions.

$$\frac{\gamma}{c_t^T} = \lambda \tag{A.4}$$

$$\frac{1-\gamma}{c_t^N} = \frac{\lambda}{e_t} \tag{A.5}$$

$$\frac{\alpha}{m_t} = \lambda i_t \tag{A.6}$$

$$\alpha A_t l_t^{\alpha-1} = \frac{\beta B_t \left(1 - l_t\right)^{\beta-1}}{e_t} \tag{A.7}$$

Combining (A.4) and (A.5) yields an expression for the real exchange rate that must hold at all points in time.

$$e_t = \frac{c_t^N}{c_t^T}\left(\frac{\gamma}{1-\gamma}\right) \tag{A.8}$$

Combining (A.4) and (A.6) yields an expression for real money demand in terms of the traded good,

$$m_t = \frac{\alpha}{\gamma}\frac{c_t^T}{i_t}. \tag{A.9.a}$$

Combining (A.5) and (A.6) yields real money demand in terms of the non-traded good

$$n_t = \frac{\alpha}{1-\gamma}\frac{c_t^N}{i_t} \tag{A.9.b}$$

where $n\left(\equiv M/P_H\right)$.

Equilibrium Conditions

Interest parity requires

$$i_t = i_t^* + \varepsilon_t \tag{A.10}$$

where i represents the domestic nominal interest rate, i^* the foreign (world) interest rate and, ε, the depreciation rate of domestic currency.

Market clearing in the non-traded goods market implies

$$y_t^N = c_t^N \qquad \textit{for all } t. \tag{A.11}$$

In a perfect foresight equilibrium, traded and non-traded good consumption are both constant. That traded consumption is constant follows from (A.4). To show that home good consumption is constant, we must first show that the real exchange rate is also constant in equilibrium. Suppose it is not constant: Consider the case of an increase in the real exchange rate. By totally differentiating (A.7) it can be shown that an increase in the real exchange rate generates an increase in labor in the traded sector, l.[2] By (A.3.b), an increase in l leads to a contraction in non-traded good output which, by (A.11), leads to a fall in non-traded consumption. But, by (A.5), this leads to a contradiction since we can't have an increase in the real exchange rate and a fall in non-traded consumption. Similar logic holds for a decrease in the real exchange rate. Therefore it is constant. In equilibrium, $e_t = \overline{e}$, $y_t^N = \overline{y}$, and $c_t^N = \overline{c}^N$ by (A.5).

Government revenue from money creation is given back to the individuals via the government transfer, τ, and this leads to the following overall budget constraint (Balance of Payments equation) for the economy as a whole:

$$\dot{k}_t = rk_t + y_t^T + f_t - c_t^T \tag{A.12}$$

where k is the sum of asset holdings of individuals, a, plus official asset holdings (reserves) of the government (central bank), h, therefore $k \equiv a + h$.

Solving (A.12) in terms of the traded good and integrating forward yields an expression for the traded good consumption in equilibrium.

$$\overline{c}^T = rk_0 + \overline{y}^T + \overline{f}, \tag{A.13}$$

which says traded good consumption depends on the flow of returns from the initial asset holdings, the constant flow of remittances and traded good production. It is known to be constant (piecewise linear) by (A.4).

---

2 $$\frac{dl}{de} = -\frac{1}{\frac{\beta}{\alpha}\frac{B}{A}(\beta-1)(1-l)^{\beta-2}l^{1-\alpha} - \frac{\beta}{\alpha}\frac{B}{A}(1-\alpha)(1-l)^{\beta-2}l^{-\alpha}} > 0$$

Combining (A.13) and (A.11) with (A.8) yields an expression for the equilibrium real exchange rate.

$$\bar{e} = \frac{y^N}{rk_0 + y^T + f} \cdot \frac{\gamma}{1-\gamma} \qquad \text{(A.14)}$$

where y^N and y^T are given by (A.3.a) and (A.3.b).

Remittances and The Real Economy: Effects of the "Dutch disease"

In our model, remittances will always cause a real appreciation and resultant resource allocation á la the "Dutch disease", independent of the economy's monetary regime. To see this, first equate (A.7) and (A.14) to substitute out the real exchange rate, use (A.3.a) and (A.3.b), and rearrange to obtain an expression in terms of labor, remittances, and parameters.[3] Totally differentiating this shows that increasing remittances, *f*, requires a fall in the amount of labor employed in the traded good sector, *l*. This is the so-called "resource movement effect".

$$\frac{dl}{df} = \frac{1}{\left(\frac{\gamma}{1-\gamma}\frac{\alpha}{\beta}AB\right)(\alpha-1)l_t^{\alpha-2} - \left(\frac{\gamma}{1-\gamma}\frac{\alpha}{\beta}AB\right)\alpha l_t^{\alpha-1} - A\alpha l_t^{\alpha-1}} < 0 \qquad \text{(A.15)}$$

To see the "spending effect" (i.e., the effect on the real exchange rate), rewrite optimality condition (A.7) in terms of the real exchange rate[4] and differentiate with respect to traded good sector labor.

$$\frac{de}{dl} = -\frac{\beta}{\alpha}(\beta-1)(1-l)^{\beta-2}l^{1-\alpha} + \frac{\beta}{\alpha}(1-\alpha)(1-l)^{\beta-1}l^{-\alpha} > 0 \qquad \text{(A.16)}$$

which says that an increase in labor to the traded sector increases the real exchange rate.

Combining (A.15) and (A.16), gives the full "Dutch disease" effect.

$$\frac{de}{df} = \frac{de}{dl}\frac{dl}{df} < 0\,. \qquad \text{(A.17)}$$

That is, an increase in remittances always generates a fall in the real exchange rate in this economy. Notice that this result follows independent of the exchange rate adopted because money is a veil here.

---

[3] For reviewers: $\left(\frac{\gamma}{1-\gamma}\frac{\alpha}{\beta}AB\right)l_t^{\alpha-1} - \left(\frac{\gamma}{1-\gamma}\frac{\alpha}{\beta}AB\right)l_t^{\alpha} - ra_0 - Al_t^{\alpha} - f_t = 0$

[4] For reviewers: $e = \frac{\beta}{\alpha}\frac{B(1-l)^{\beta-1}}{An^{\alpha-1}}$

Furthermore, the income effect from increased wealth in the form of remittance inflows leads to an increase in consumption of both goods. By (A.17) and (A.8) the final change in both levels of consumption must be such that traded good consumption increases by more than home good consumption. Analytically, use (A.3.b) in equilibrium condition (A.11) and differentiate with respect to remittances,

$$\frac{dc_t^N}{df} = -\beta B\left(1-l\right)^{\beta-1}\frac{dl}{df} > 0. \tag{A.18}$$

Again, since c^N increases yet the real exchange rate falls, by (A.17), c^T must also increase and by more than the increase in c^N. From (13) with (3.a) and using (17) to sign,

$$\frac{dc_t^T}{df} = \alpha A l^{\alpha-1}\frac{dl}{df} + 1 > 0 \tag{A.19}$$

which imposes a restriction on the magnitude of the resource allocation effect in the traded sector such that $\left|\alpha A l^{\alpha-1}\frac{dl}{df}\right| < 1$.

Monetary Regimes and Economic Dynamics

To generate dynamics in this model, we assume that non-traded good prices adjust according to a Calvo-type pricing mechanism.

$$\dot{\pi}_t = -\theta(c_t^N - \overline{y}^N) \qquad \theta > 0 \tag{A.20}$$

where $\overline{y}^N$ is the steady state level of non-traded good production and θ is a constant parameter.

Fixed Exchange Rate Regime

By $\dot{m} = m(\mu_t - \overline{\varepsilon})$ and $\dot{m} = 0$ it follows that $\mu = \overline{\varepsilon}$ in steady state where μ is the rate of nominal money supply growth. Likewise, constant currency depreciation implies by interest parity that the nominal interest rate is constant, $\overline{i} = i^* + \overline{\varepsilon}$, in this regime. Notice that while money growth can change outside of steady state, the nominal interest rate is constant at all times given the assumption of constant foreign inflation and real world interest rate.

The economy's behavior is thus governed by the following two differential equations.

$$\dot{\pi}_t = \theta\left(\overline{y}^N\left(\overline{l}\right) - \frac{1-\gamma}{\gamma}\cdot e_t\overline{c}^T\right) \tag{A.21}$$

$$\dot{e} = e_t(\bar{\varepsilon} - \pi_t) \quad \text{(A. 22)}$$

where $\bar{y}^N(\bar{l}) = B(1-\bar{l})^{\beta}$. Thus, changes in the steady state employment allocation change the steady state level of non-traded good production.

Linearizing around steady state, this system can be expressed as

$$\begin{bmatrix} \dot{\pi}_t \\ \dot{e}_t \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1-\gamma}{\gamma}\bar{c}^T\theta \\ -\bar{e} & 0 \end{bmatrix} \begin{bmatrix} \pi_t - \bar{\pi} \\ e_t - \bar{e} \end{bmatrix}. \quad \text{(A.23)}$$

Given the negative determinant, $\det = -\frac{\bar{e}(1-\gamma)}{\gamma}\bar{c}^T\theta < 0$, and zero trace, the system has one negative root (for state variable *e*) and one positive root (for control variable π), thus exhibiting saddle path dynamics.

When the flow of remittances increases, the new steady state must exhibit a lower real exchange rate, by (17), higher non-traded good output, by (11) and (18), as well as higher traded and non-traded good consumption, by (18) and (19). Thus, on impact, traded good consumption and the steady state level of non-traded good production in equation (21) both jump to their new, higher levels. Since the real exchange rate will be lower in the new steady state, we know that traded good consumption changes by more than the steady state non-traded good output. Given that we started in steady state and the real exchange rate, *e*, is a predetermined, it follows that the right hand side of (21) turns negative upon impact of the shock to remittances. For this to hold and for the real exchange rate to reach it's new, lower steady state level, the inflation rate must increase upon impact to generate the necessary dynamics according to (22). This is represented in the phase diagram in Figure 1 in the paper.

Assume an initial trade balance, zero remittance inflows and no pre-existing asset holdings. Then, in the initial steady state, $\bar{c}^T = \bar{y}^T$, by equation (A.13). When the increase in remittances occurs, individuals re-optimize. Integrating (A.12) from the instant of the shock (date *t = 0*) forward yields the appropriate expression for the new steady state level of traded good consumption,

$$\bar{\bar{c}}^T = r\int_0^{\infty} y_t^T e^{-rt}dt + \bar{f} \quad \text{(A.24)}$$

where double over strikes are used to distinguish this from $\bar{c}^T$ in (A.13). By (A.3.a) and (A.16), traded good production is a function of the real exchange rate. In the new steady state, the real exchange rate and traded good production are lower by (A.16), (A.17), and (A.19). Since the real exchange rate is a predetermined variable under the fixed regime, so is traded good production. Thus, upon impact, traded consumption jumps upward to its new stead state level. This level accounts for the fact that traded good production falls during transition and remains at a permanently lower level. We can conclude from this that the increase in remittances must exceed the discounted value of the decline in traded production. It also follows that the initial jump in traded consumption is less than the increase in remittances. Finally, in the new steady state, the inflow of remittances must equal the difference between traded consumption and production plus the interest payments on debt acquired during transition such that the current account returns to balance.

Flexible Exchange Rate

By $\dot{m} = m(\bar{\mu} - \varepsilon_t)$ and $\dot{m} = 0$ it follows that $\bar{\mu} = \varepsilon$ in steady state. Likewise, constant currency depreciation implies by interest parity that the nominal interest rate is constant, $\bar{i} = i^* + \bar{\varepsilon}$, in steady state. Notice that while money growth is assumed fixed everywhere both the interest rate and the rate of depreciation can change outside of steady state, unlike the FIX regime case.

The system's dynamics under a FLEX are best captured in terms of real money balances in terms of the non-traded good, $n\left(\equiv M / P^N\right)$, and the non-traded good inflation rate, π. Under a FLEX, n is a predetermined variable since M is exogenous and constant and P^N is predetermined. π remains a control variable. Differentiating the definition of real money balances with respect to time yields

$$\dot{n}_t = n_t(\bar{\mu} - \pi_t). \tag{A.25}$$

Using (A.8) and (A.9.a), substitute into (A.20) for c^N and rearrange to obtain

$$\dot{\pi}_t = \theta\left(\bar{y}^N\left(\bar{l}\right) - \frac{1-\gamma}{\alpha} i_t n_t\right) \tag{A.26}$$

where, again, $\bar{y}^N\left(\bar{l}\right) = B\left(1-\bar{l}\right)^\beta$.

Linearizing around steady state,

$$\begin{bmatrix} \dot{\pi}_t \\ \dot{n}_t \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1-\gamma}{\alpha}\bar{i}\,\theta \\ -\bar{n} & 0 \end{bmatrix} \begin{bmatrix} \pi_t - \bar{\pi} \\ n_t - \bar{n} \end{bmatrix} \tag{A.27}$$

This system has a negative determinant, $\det = -\bar{n}\frac{1-\gamma}{\alpha}\bar{c}^T\theta < 0$, and a trace of zero. There is one positive (for control variable π) and one negative (for state variable n) root.

As before, due to the increase in remittances, the new steady state must exhibit a lower real exchange rate, e, higher non-traded good output, $\bar{y}^N$, as well as higher traded and non-traded good consumption, $\bar{c}^T$ and $\bar{c}^N$, respectively. Real money balances in terms of the non-traded good, n, is predetermined and thus constant on impact. Likewise, i remains unchanged since under the FLEX, the nominal exchange rate jumps to its new level on impact to maintain equilibrium in the money market described by (A.9.a). The steady state level of non-traded good production is not constant, however, and jumps on impact to its new, higher level. The result is that, the right hand side of (A.26) turns positive upon impact. For this to hold and for the real exchange rate to reach it's new, lower steady state level, the inflation rate must decrease upon impact to generate the necessary dynamics according to (A.25). This is represented in the phase diagram in Figure 2 in the paper.

Since the nominal stock of money is constant under a FLEX (row 4), the nominal exchange rate must fall upon impact to maintain real money market equilibrium, equation (9.a). The non-traded good price level is predetermined and thus unchanged initially and, given the initial drop in inflation, rises slowly at first but more rapidly as the economy approaches the new steady state. As a result, the real exchange rate drops initially and rises to its new, lower steady state level. That is, it undershoots it's new steady state level (row 1). By (3.a) and (16), traded good production is a function of the real exchange rate and follows a similar path. Given the increase in traded good consumption on impact, the decline then subsequent rise in traded production implies that the trade balance is initially a large deficit but one that closes as the level of production increases during the transition (row 9). Again, in the new steady state the economy runs a trade deficit where the remittance inflow finances the trade deficit and interest payments on accumulated debt such that the current account is in balance.

Mathematical Appendix Solving the Model in State – Costate Space

These are the math notes for an earlier version of the model where we had considered two versions of a FLEX: one with a fixed money supply and one with an inflation targeting interest rate rule. We later dropped the inflation targeting version from the paper because the simple FIX and simple FLEX got all the main points across theoretically and could be easily represented in a 2-dimensional phase diagram. Adding the interest rate rule added a second state variable (the interest rate) and thus took us to 3D space. This would require us to use 2 phase diagrams for the regime or plots all the time paths. We felt the marginal benefit to this rather large marginal cost in presentation wasn't worthwhile and would only raise more questions needlessly.

The notation here is slightly different than in the paper because it's an earlier version, but nothing in the model is different. Here's a little table to convert notation.

Variables in Final Paper	**Variables in This Appendix Model**
a = net assets	b = gross bonds/assets
e = real exchange rate	ε = real exchange rate
ε = rate of currency depreciation	$\hat{E}$ = rate of currency depreciation
τ = transfer from government ("tax" or "subsidy")	s = transfer from government ("tax" or "subsidy")
subcript H (on P_H, y_H and c_H) denotes "home" good (i.e., produced and consumed in the home country)	subscript N (on P_N, y_N, and c_N) = "nontraded" good (i.e., produced and consumed in the home country)

Lastly, the paper uses a condensed version of the budget constraint here. The budget constraint here is

$$\dot{b} = y_T + \frac{y_H}{\varepsilon} + rb + s + f - c_T - \frac{c_H}{\varepsilon} - \dot{m} - \hat{E}m$$

To rearrange the budget constraint, first rearrange the budget constraint as follows:

$$\dot{b} + \dot{m} = y_T + \frac{y_H}{\varepsilon} + \left(rb - \hat{E}m\right) + s + f - c_T - \frac{c_H}{\varepsilon}$$

Next, define net assets holdings as $a = b + m$ and note that this implies $\dot{a} = \dot{b} + \dot{m}$. Substitute this in for b, m and their time derivatives, rearrange interest parity ($i = r + \hat{E}$) as $\hat{E} = i - r$ and substitute this in as well:

$$\dot{a} = y_T + \frac{y_H}{\varepsilon} + \left(rb - rm - im\right) + s + f - c_T - \frac{c_H}{\varepsilon}$$

and, finally,

$$\dot{a} = y_T + \frac{y_H}{\varepsilon} + rb + s + f - c_T - \frac{c_H}{\varepsilon} - im$$

which is the budget constraint used in the paper.

Individuals solve the following problem.

$$\max_{m, c_T, c_H} \int_0^\infty \left[u\left(c_T, c_H\right) + v\left(m\right)\right] e^{-\rho t} dt$$

subject to

$$\dot{b} = y_T + \frac{y_H}{\varepsilon} + rb + s + f - c_T - \frac{c_H}{\varepsilon} - \dot{m} - \hat{E}m$$

Modified Hamiltonian in Current Value Form

$$\tilde{H} = u\left(c_T, c_H\right) + v\left(m\right) + \psi \dot{b} + \left(\dot{\psi} - \rho\psi\right) b$$

where $\psi \equiv e^{\rho t}\lambda$

FOC 1: $u_T\left(c_T, c_H\right) - \psi = 0$

FOC 2: $u_H\left(c_T, c_H\right)\varepsilon - \psi = 0$

FOC 3: $v_m\left(m\right) - \hat{E}\psi = 0$

FOC 4: $\dot{\psi} - \rho\psi + r\psi = 0$

Overall resource constraint: $\dot{b} = rb + f + y_T - c_T$

Next we need to find the implicit functional relationships between these variables of interest. I first do it in general, then, off to the right, with our explicit functional forms:

$$u(c_T, c_H) = c_T^{\gamma} c_H^{1-\gamma} \qquad \& \qquad v(m) = \ln(m)$$

From FOC 1: $u_T(c_T, c_H) - \psi = 0 \qquad \Rightarrow \qquad c_T = \left(\frac{\psi}{\gamma} c_H^{\gamma-1}\right)^{\frac{1}{\gamma-1}}$

$$\frac{dc_T}{d\psi} = \frac{1}{u_{TT}} < 0 \qquad \Rightarrow \qquad \frac{dc_T}{d\psi} = \frac{1}{\gamma-1}\left(\frac{\psi}{\gamma} c_H^{\gamma-1}\right)^{\frac{1}{\gamma-1}-1} \frac{c_H^{\gamma-1}}{\gamma} < 0 \qquad \because \gamma - 1 < 0$$

$$\frac{dc_T}{dc_H} = -\frac{u_{TH}}{u_{TT}} > 0 \qquad \Rightarrow \qquad \frac{dc_T}{d\psi} = \left(\frac{\psi}{\gamma} c_H^{\gamma-1}\right)^{\frac{1}{\gamma-1}-1} \frac{\psi c_H^{\gamma-2}}{\gamma} > 0$$

$$\text{So, } c_T = \left(\frac{\psi}{\gamma} c_H^{\gamma-1}\right)^{\frac{1}{\gamma-1}} \qquad \Rightarrow \qquad c_T = c_T\left(\overbrace{\psi}^{-}, \overbrace{c_H}^{+}\right)$$

From FOC 2: $u_H(c_T, c_H)\varepsilon - \psi = 0 \qquad \Rightarrow \qquad c_H = \left(\frac{\psi}{1-\gamma}\frac{1}{\varepsilon} c_T^{-\gamma}\right)^{-\frac{1}{\gamma}}$

$$\frac{dc_H}{d\psi} = \frac{1}{u_{TT}} < 0 \qquad \Rightarrow \qquad \frac{dc_H}{d\psi} = -\frac{1}{\gamma}\left(\frac{\psi}{1-\gamma}\frac{1}{\varepsilon} c_T^{-\gamma}\right)^{-\frac{1}{\gamma}-1} \frac{1}{\varepsilon}\frac{c_T^{-\gamma}}{1-\gamma} < 0$$

$$\frac{dc_H}{dc_T} = -\frac{u_{TH}}{u_{HH}} > 0 \qquad \Rightarrow \qquad \frac{dc_H}{dc_T} = \left(\frac{\psi}{1-\gamma}\frac{1}{\varepsilon} c_T^{-\gamma}\right)^{-\frac{1}{\gamma}-1} \frac{1}{\varepsilon}\frac{c_T^{-\gamma-1}}{1-\gamma} > 0$$

$$\frac{dc_H}{d\varepsilon} = -\frac{u_H}{u_{HH}} > 0 \qquad \Rightarrow \qquad \frac{dc_H}{d\varepsilon} = -\frac{1}{\gamma}\left(\frac{\psi}{1-\gamma}\frac{1}{\varepsilon} c_T^{-\gamma}\right)^{-\frac{1}{\gamma}-1} \left(-\frac{1}{\varepsilon^2}\frac{c_T^{-\gamma}}{1-\gamma}\right) > 0$$

$$\text{So, } c_H = \left(\frac{\psi}{1-\gamma}\frac{1}{\varepsilon} c_T^{-\gamma}\right)^{-\frac{1}{\gamma}} \qquad \Rightarrow \qquad c_H = c_H\left(\overbrace{\psi}^{-}, \overbrace{c_T}^{+}, \overbrace{\varepsilon}^{+}\right)$$

From FOC 3: $v_m(m) - \hat{E}\psi = 0$ and $i = r + \hat{E}$ we have $v_m(m) - (i - r)\psi = 0$

$$m = \frac{1}{(i-r)\psi} \qquad \Leftrightarrow \qquad \psi = \frac{1}{m(i-r)} \qquad \Leftrightarrow \qquad i = \frac{1}{\psi m} + r$$

$$\frac{d\psi}{dm} = \frac{v_{mm}}{i-r} < 0 \qquad \Rightarrow \qquad \frac{d\psi}{dm} = -\frac{1}{m^2\left(i-r\right)} < 0$$

$$\frac{d\psi}{di} = \frac{-\psi}{i-r} < 0 \qquad \Rightarrow \qquad \frac{d\psi}{dm} = -\frac{1}{i^2 m} < 0$$

So, $\psi = \psi\left(\overbrace{m}^{-}, \overbrace{i}^{-}\right) \quad \Leftrightarrow \quad \psi = \psi\left(\overbrace{m}^{-}, \overbrace{\hat{E}}^{-}\right)$

And, we also get

$$\frac{di}{dm} = \frac{v_{mm}}{\psi} < 0 \qquad \Rightarrow \qquad \frac{di}{dm} = -\frac{1}{m^2\psi} < 0$$

$$\frac{di}{dr} = \frac{\psi}{\psi} = 1 > 0$$

$$\frac{di}{d\psi} = \frac{-\left(i-r\right)}{\psi} < 0 \qquad \Rightarrow \qquad \frac{di}{d\psi} = -\frac{1}{\psi^2 m} < 0$$

So, $i = i\left(\overbrace{\psi}^{-}, \overbrace{m}^{-}\right)$

Dynamic Systems

Relations among variables

$$c_T = c_T(\overbrace{\psi}^{-}, \overbrace{c_H}^{+})$$

$$c_H = c_H(\overbrace{\psi}^{-}, \overbrace{c_T}^{+}, \overbrace{\varepsilon}^{+})$$

$$\psi = \psi(\overbrace{m}^{-}, \overbrace{i}^{+}) \quad or \quad \psi = \psi(\overbrace{m}^{-}, \overbrace{\hat{E}}^{+})$$

$$i = i(\overbrace{\psi}^{-}, \overbrace{m}^{-})$$

$$i = r + \hat{E}$$

Sluggish Home Prices

FLEX

Control variables: ψ , π_H

State variables: $n\left(\equiv M/P_H\right)$ Recall that m = n/ε

$$\hat{\psi} = \rho + \hat{E} - i\left(\psi, n\right)$$

$$\dot{\pi}_H = \theta y_H - \theta c_H\left(\psi\left(n, i\right), \varepsilon\right)$$

$$\hat{n} = \mu - \pi_H$$

$$\begin{bmatrix} \dot{\psi} \\ \dot{\pi}_H \\ \hat{n} \end{bmatrix} = \begin{bmatrix} -\frac{\partial i}{\partial \psi} & 0 & -\frac{\partial i}{\partial n} \\ -\theta \frac{\partial c_H}{\partial \psi} & 0 & -\theta \frac{\partial c_H}{\partial n} \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} \psi - \psi^* \\ \pi_H - \pi_H^* \\ n - n^* \end{bmatrix}$$

$$|J| = -\frac{\partial i}{\partial \psi} \begin{vmatrix} 0 & -\theta \frac{\partial c_H}{\partial n} \\ -1 & 0 \end{vmatrix} - \frac{\partial i}{\partial n} \begin{vmatrix} -\theta \frac{\partial c_H}{\partial \psi} & 0 \\ 0 & -1 \end{vmatrix} = \underbrace{\underbrace{\frac{\partial i}{\partial \psi}}_{-} \left(\theta \underbrace{\frac{\partial c_H}{\partial n}}_{+} \right)}_{-} \underbrace{- \underbrace{\underbrace{\frac{\partial i}{\partial n}}_{-} \left(\theta \underbrace{\frac{\partial c_H}{\partial \psi}}_{-} \right)}_{+}}_{-} < 0$$

$$Tr(J) = -\frac{\partial i}{\partial \psi} > 0$$

This says one negative and two positive roots. 1 negative per state. OK.